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THEOREM ABOUT THE CONJUGACY REPRESENTATION OF S_n

BY
AVITAL FRUMKIN[†]
Department of Theoretical Mathematics, The Weizmann Institute of Science, Rehovot 76100, Israel

ABSTRACT
Every group has two natural representations on itself, the regular representation and the conjugacy representation. We know everything about the construction of the regular representation, but we know very little about the conjugacy representation (for uncommutative groups). In this paper we will see that every irreducible complex character of S_n $(n > 2)$ is a constituent of conjugacy character of S_n.

Introduction

Let S_n be the symmetric group of degree n and $\mathbf{Q}S_n$ the group ring of S_n over the rational number field $\mathbf{Q}$. We denote the left regular module of S_n by LRMS_n and the conjugacy module by COMS_n.

By the conjugacy module we mean the vector space $\mathbf{Q}S_n$ and the action $\sigma^*\eta = \sigma\eta\sigma^{-1}$ for every $\sigma, \eta \in S_n$. (This action is extended linearly to $\mathbf{Q}S_n$.) The representation afforded by the conjugacy module is called the conjugacy representation and the corresponding character is called the conjugacy character. Our purpose is to prove:

THEOREM 1. *Every irreducible complex character of S_n is a constituent of the conjugacy character of S_n.*

Note that in view of [3] Theorem 1 or [2] Lemma 1.1 this is equivalent to the theorem below.

THEOREM 2. *Let x be any ordinary character of S_n and C the set of conjugacy classes of S_n. For each $c \in C$ we choose a representative $c' \in c$. Then $\Sigma_{c\in C}\, x(c') > 0$.*

[†] *Current address*: Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem, Israel.

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1. Preliminaries and reduction

We will show that every irreducible character of S_n is a constituent of the conjugacy character. Generally we follow the notation of [1].

The action of S_n on the set $\{1,2,\ldots,n\}$ is from the right.

We denote by $C_{S_n}(\rho)$ the centralizer of the permutation ρ in S_n.

LEMMA 1. *If $\sigma \in C_{S_n}(\rho)$, then σ permutes the cycles (orbits) of ρ.*

PROOF. It is a well-known fact. (See e.g. [4].)

We need some facts on the representations of S_n. These are taken from [1] and described briefly below.

For every Young diagram D, we let $R(D)$ be the group of row permutations and $C(D)$ the group of column permutations. Define

$$e(D) = \sum_{\substack{p \in R(D) \\ q \in C(D)}} pq \operatorname{sig}(q)$$

where $\operatorname{sig}(q)$ is the sign of the permutation q.

Then $e(D) \in \mathbf{Q}S_n$ and $(\mathbf{Q}S_n)e(D)$ is a minimal left ideal in LRMS_n. Every irreducible representation of S_n is afforded by $(\mathbf{Q}S_n)e(D)$ for some Young diagram D.

Next we describe homomorphisms from LRMS_n to COMS_n.

Let $\phi : \mathrm{LRM}S_n \to \mathrm{COM}S_n$ be a homomorphism. Then for each $\sigma \in S_n$ we have that $\phi(\sigma) = \sigma\phi(I)\sigma^{-1}$ where I is the identity permutation. Therefore ϕ is determined by $\phi(I)$. On the other hand, for every $\rho \in S_n$ there is a homomorphism $\phi_\rho : \mathrm{LRM}S_n \to \mathrm{COM}S_n$ defined by the formula $\phi_\rho(\sigma) = \sigma\rho\sigma^{-1}$ (and extended linearly) for each $\sigma \in S_n$.

PROPOSITION. *In order to prove Theorem 1 it suffices to construct for every Young diagram D a permutation ρ_D such that $\Sigma_{pq \in Y_D} \operatorname{sig}(q) \neq 0$ where $Y_D = C_{S_n}(\rho_D) \cap R(D) \cdot C(D)$.*

PROOF. Suppose such a ρ_D has been constructed, then

$$(\phi_{\rho_D})(e(D)) = \sum_{\substack{p \in R(D) \\ q \in C(D)}} pq\rho_D q^{-1} p^{-1} \operatorname{sig}(q).$$

Let $(\ \ )_\rho$ be the ρ-th coefficient of $(\ \ )$. Then

$$\phi_{\rho_D}(e(D))_{\rho_D} = \sum_{\substack{pq\rho_D q^{-1} p^{-1} = \rho_D \\ p \in R(D),\, q \in C(D)}} \operatorname{sig}(q) = \sum_{pq \in Y_D} \operatorname{sig}(q).$$

By assumption $\phi_{\rho_D}(e(D)) \neq 0$ and so $(\mathbf{Q}S_n)e(D) \not\subset \ker(\phi_{\rho_D})$. By Schur's Lemma we conclude that $\phi_{\rho_D}((\mathbf{Q}S_n)e(D)) \cong (\mathbf{Q}S_n)e(D)$. □

We now turn to the construction of the desirable ρ_D for each Young diagram D. We will first construct ρ for two classes of basic diagrams, including five special cases in all. We will then utilize these cases in the construction of ρ_D for arbitrary D.

2. Construction of ρ_D for particular cases

Class A

Let D be a rectangular diagram

$$D = \begin{cases} 1 & 2F & 2F+1 & \cdots & LF \\ 2 & 2F-1 & \cdot & & \cdot \\ \vdots & & \vdots & & \vdots \\ F & F+1 & 3F & & LF-F \end{cases}$$

Case (i). Assume that L is an even number. We define

$$\rho_D = (1,2F,2F+1,\ldots,LF)(2,2F-1,2F+2,\ldots,LF-1)\cdots (F,F+1,3F,\ldots,LF-F).$$

ρ_D is a permutation for which D's rows are its cycles.

The Calculation of $\Sigma_{pq\in Y_D} \operatorname{sig} q$

If $\eta \in Y_D$ then $\eta \in C_{S_n}(\rho_D)$; η acts on D's rows (by Lemma 1 of Section 1). Therefore there exist $p \in R(D)$, $q \in C(D)$ such that $\eta = pq$ with q acting on each column in the same way. Defining $q\,|_i$ as the restriction of q to the i-th column we get $q = \Pi_i\,(q\,|_i)$ and

$$\operatorname{sig}(q) = \prod_i \operatorname{sig}(q\,|_i) = (\operatorname{sig} q\,|_i)^L = 1$$

since L is even. Therefore

$$\sum_{pq\in Y_D} \operatorname{sig}(q) = |C_{S_n}(\rho_D)| \geqq 1. \qquad \square$$

Case (ii). Assume now that L is an odd number greater than 1.

We define $\rho_D = (1,2,\ldots,F,F+1,\ldots,2F,2F+1,\ldots,LF)$, i.e., a cyclic permutation. The construction of ρ_D from D is described in the following diagram:

$$D = \begin{cases} 1 & 2F & 2F+1 & \cdots & LF-1 \\ 2 & & \vdots & & \vdots \\ & & & & \\ \vdots & & \vdots & & \vdots \\ F & F+1 & 3F & \cdots & LF \end{cases}$$

$$\rho_D = \begin{cases} 1 & 2F & 2F+1 & \cdots & LF-F \\ \downarrow & \uparrow & \downarrow & & \downarrow \\ 2 & \cdot & \cdot & & \cdot \\ \vdots & \vdots & \vdots & & \vdots \\ \downarrow & \uparrow & \downarrow & & \downarrow \\ F & F+1 & 3F & \cdots & LF \end{cases}$$

Calculation of $\Sigma_{pq\in Y_D}\operatorname{sig}(q)$

ρ_D is cyclic, therefore $C_{S_n}(\rho_D)=\{\rho_D^k : 0\leqq k\leqq LF-1\}$. The following lemma characterizes the elements of Y_D for this case.

LEMMA. *$\rho_D^k \in Y_D$ iff $F \mid k$.*

PROOF. Assume $k = lF+S$ with $0<S<F$, $S\in N$, $l\in N$. We obtain

$$(F)\rho_D^k = \begin{cases} F+lF+S, & l<L-1, \\ S, & l=L-1; \end{cases}$$

$$(F+1)\rho_D^k = \begin{cases} F+lF+S+1, & l<L-1, \\ S+1, & l=L-1. \end{cases}$$

We can see that F and $F+1$ are transferred to the same column. Remember, however, that F and $F+1$ belong to the last row of D. This implies that $\rho^k \notin R(D)C(D)$. (See e.g. [1].) This, in turn, implies that $\rho_D^k \notin Y_D$. If $S=0$ then ρ_D^k acts on the column of D so $\rho_D^k \in R(D)C(D)$ (by [1]). □

Applying the lemma we get

$$\sum_{pq\in Y_D}\operatorname{sig}(q) = \sum_{pq=\rho_D^{lF},\,0\leqq l\leqq L-1}\operatorname{sig}(q).$$

The last sum isn't zero because it is the sum of an odd number of ± 1. □

Case (iii). Let D be a rectangular diagram such that $L=1$ and F is odd.

$$D = \left\{ \begin{array}{c} 1 \\ 2 \\ \vdots \\ F \end{array} \right.$$

We define $\rho_D = (1,\ldots,F)$. ρ_D is a one cycle.

Calculation of $\Sigma_{pq \in Y_D} \operatorname{sig}(q)$

ρ_D is a one cycle, therefore $C_{S_n}(\rho_D) = \{\rho_D^k : 0 \leqq k \leqq F-1\}$. For every k, $\rho_D^k \in C(D)$. Therefore $Y_D = C_{S_n}(\rho_D)$.

For every k, $\operatorname{sig}(\rho_D^k) = 1$, since F is odd, so $\Sigma_{pq \in Y_D} \operatorname{sig}(q) = F > 0$. □

Case (*iv*). Again let D be rectangular and $L = 1$ but now F is even and greater than 2. We define $\rho_D = (1,2,\ldots,F-1)(F)$. ρ_D is the multiplication of a non-trivial cycle and the fixed point (F).

$C_{S_n}(\rho_D) = \{\rho_D^k : 0 \leqq k \leqq F-2\}$ and again we have for every k, $\rho_D^k \in C(D)$, so that $Y_D = C_{S_n}(\rho_D)$. As F is even, therefore $\operatorname{sig}(\rho_D^k) = 1$ for every k and $\Sigma_{pq \in Y_D} \operatorname{sig}(q) = F - 1 > 0$. □

Class B

Class B includes one case only. Let D be the diagram illustrated below:

$$\begin{array}{|c|c|} \hline 3 & 4 \ \cdots \ L \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \qquad L \geqq 4.$$

We define ρ_D to be the cycle $(1,2,\ldots,L)$. ρ_D is a cylic permutation, therefore $C_{S_n}(\rho_D) = \{\rho_D^k : 0 \leqq k \leqq L-1\}$. Assume $\rho_D^k \in Y_D$, then $\rho_D^k \in R(D)C(D)$ ($\rho_D^k = pq$, $p \in R(D)$, $q \in C(D)$). ρ_D^k conserves 1,2 in their column (because the action is from the right) so $k = 0$ or $k = 1$ (see drawing above).

If $k = 0$ then $\rho_D^0 = I$. If $k = 1$ then $\rho_D^1 = \rho_D = (3,4,\ldots,L)(1,2,3)$, $(3,4,\ldots,l) \in R(D)$ and $(1,2,3) \in C(D)$. therefore

$$\sum_{pq \in Y_D} \operatorname{sig}(q) = \operatorname{sig}(I) + \operatorname{sig}(1,2,3) = 2.$$

3. The general case

Let D be an arbitrary Young diagram and let D_i denote the rectangle in D with length i, i.e., D_i is the rectangle consisting of the rows of length i in D.

Let $\{D_i\} = \{\mu : \mu$ is a digit in $D_i\}$. We assume first that the height of D_1 doesn't equal two (the number of D_1's rows $\neq 2$).

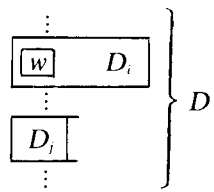
We define $\rho_D = \Pi_i \rho_{D_i}$ (the ρ_{D_i} have been defined in section 2 under Class A).

LEMMA. *If $\eta \in Y_D$ then $\{D_i\}\eta = \{D_i\}$ for every i.*

IMPORTANT REMARK. In all cases of section 2 every two digits in the same row of D belong to the same cycle in ρ_D.

PROOF OF THE LEMMA. If $\eta \in C_{S_n}(\rho_D)$ then η acts on the cycles of ρ_D (Lemma 1 of section 1).

Let w be a digit in D_i. Assume that $(w)\eta \in \{D_j\}$ where $j < i$, then the whole cycle to which w belongs is transferred to D_j by η. Then (by the remark) the row in which w appears is transferred to D_j.



We denote the row of w by $A(w)$. Let $p \in R(D)$, $q \in C(D)$. In this notation, the right digit in $\{A(w)\}p$ cannot be transferred to $\{D_j\}$ by $q \in C(D)$. (See the drawing above.) Therefore $\eta \notin Y_D$. $\square$

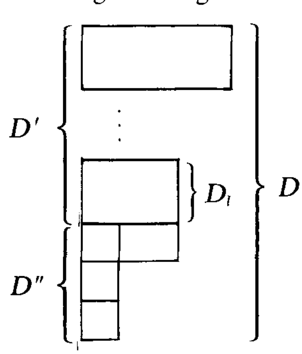
Calculation of $\Sigma_{pq \in Y_D} \operatorname{sig}(q)$

From the last lemma it is apparent that $Y_D = \Pi_i Y_{D_i}$. Consequently

$$\sum_{pq \in Y_D} \operatorname{sig}(q) = \sum_{pq \in \Pi_i Y_{D_i}} \operatorname{sig}(q) = \prod_i \sum_{pq \in Y_{D_i}} \operatorname{sig}(q).$$

Since we have proven that for every i, $\Sigma_{pq \in Y_{D_i}} \operatorname{sig}(q) \neq 0$, we have $\Sigma_{pq \in Y_D} \operatorname{sig}(q) \neq 0$.

Assume now that the height of D_1 equals 2. Clearly we may assume that $n > 2$. The following drawing characterizes D:



Define D', D'' by $D' = D - D''$;

D'' are the last 3 rows of D.

We define $\rho_D = \rho_{D'} \cdot \rho_{D''}$, where $\rho_{D'}$ is defined in the first class of this section and $\rho_{D''}$ is defined in part B of section 2.

LEMMA. *If $\eta \in Y_D$ then $\{D'_i\}\eta = \{D'_i\}$.*

Using the same reasoning as before, it is clear why the rectangles which do not touch D'' are conserved by η. Denote by D_l the last rectangle in D' (l is the length of D_l). D_l touches D'' (see drawing above).

We prove now that if $\eta \in Y_D$ then $\{D_l\}\eta = \{D_l\}$. Recall $\rho_{D''}$ is a one cycle with length $(l+2)$. If l is even then ρ_D is the multiplication of cycles of length l (according to Case (i) of section 2).

Therefore ρ_{D_l} and $\rho_{D''}$ have no cycles of the same length. Together with the fact that if $\eta \in Y_D$ then $\eta \in C_{S_n}(\rho_D)$ we get that $\{D_l\}\eta \cap \{D''\} = \varnothing$ (Lemma 1 of section 1).

If l is odd then ρ_{D_l} is a one cycle with length $k \cdot l$, $k \in N$, $l \geqq 3$ (according to Class A, Case (ii) of section 2).

Hence, there is no $k \in N$ such that $k \cdot l = l + 2$, so ρ_{D_l} and $\rho_{D''}$ are cycles of different length. Again we get: if $\eta \in Y_D$ then $\{D_l\}\eta \cap \{D''\} = \varnothing$. Consequently $Y_D = Y_{D'} \cdot Y_{D''}$. □

Calculation of $\Sigma_{pq \in Y_D} \mathrm{sig}(q)$

From the last lemma we get

$$\sum_{pq \in Y_D} \mathrm{sig}(q) = \sum_{pq \in Y_{D'} \cdot Y_{D''}} \mathrm{sig}(q) = \left[\sum_{pq \in Y_{D'}} \mathrm{sig}(q)\right]\left[\sum_{pq \in Y_{D''}} \mathrm{sig}(q)\right].$$

Since $\Sigma_{pq \in Y_{D'}} \mathrm{sig}(q) \neq 0$ (the first case) and $\Sigma_{pq \in Y_{D''}} \mathrm{sig}(q) \neq 0$ (Class B of section 2), we finally get $\Sigma_{pq \in D} \mathrm{sig}(q) \neq 0$. □

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